

Infrared Fixed Points and Fixed Lines for Couplings in the Chiral Lagrangian

Mario Atance^{*(1)} and Barbara Schrempp^{†(2,1)}

⁽¹⁾*Institut für Theoretische Physik und Astrophysik, Universität Kiel, D-24118 Kiel, Germany*

⁽²⁾*Institut für Experimentelle und Angewandte Physik, Universität Kiel, D-24118 Kiel, Germany*

February 1, 2008

Abstract

In the framework of the low energy chiral Lagrangian renormalization group equations for the couplings are investigated up to order p^6 – as well for $SU(2) \times SU(2)$ as for $SU(3) \times SU(3)$ chiral symmetry. Infrared attractive fixed points for ratios of $O(p^4)$ couplings are found, which turn out to agree with the values determined from experiment in a surprisingly large number of cases. Infrared attractive fixed line solutions for $O(p^6)$ couplings in terms of $O(p^4)$ couplings and among $O(p^6)$ couplings are determined.

PACS number(s): 11.10.Hi, 12.39.Fe Keywords: Chiral Lagrangian, Infrared Fixed Points

As is well known, an appropriate extension of QCD to low energies is the chiral Lagrangian, which realizes the spontaneously broken (approximate) chiral symmetry nonlinearly in terms of the light Goldstone field degrees of freedom [1]. As in any effective field theory, the Lagrangian has infinitely many contributing operators, which may be arranged according to their importance for low energy observables in an expansion in powers of p/Λ . Here p denotes the low momentum scale of interest and Λ some momentum cut-off, above which the chiral Lagrangian ceases to be valid, with $p/\Lambda \lesssim O(1)$. The number of operators contributing to each order in this expansion is finite. Similarly, perturbation theory in the number of loops and the renormalization program can be carried out for effective field theories — even though in principle infinitely many counterterms are required. The choice of a mass-independent renormalization scheme ($\overline{\text{MS}}$, $\overline{\text{MS}}$) leads, however, to counterterms which may again be arranged in an expansion in powers of p/Λ . As a consequence,

*Work supported by the spanish FPU programme under grant PF 97 25438231. E-mail atance@theo-physik.uni-kiel.de

†E-mail schrempp@physik.uni-kiel.de

in any given order in p/Λ the number of counterterms needed to absorb the divergences is again finite [2].

Of interest are the coefficients of the operators in the chiral Lagrangian, or rather — after extraction of their dimension in form of powers of Λ — the dimensionless couplings. These couplings encode in principle the information about the QCD dynamics at higher scales. Unfortunately, in practice they are unknown. There have been efforts to estimate some of them by using different techniques (lattice calculations, large N_c limit, vector meson dominance, ...) [2]. The couplings in a given order of p/Λ may be determined by experiment; one needs as many observables as couplings to be determined. Within the framework of perturbation theory and renormalization described above, the renormalization group equations for the couplings can be determined at any fixed order in p/Λ . They have all been calculated up to $O((p/\Lambda)^4)$ – abbreviated by $O(p^4)$ – and recently up to $O(p^6)$ [3] (for earlier work we refer to references quoted in Refs. [3], [4]).

In this paper we investigate the renormalization group equations (RGE) up to $O(p^6)$ and search for infrared (IR) attractive fixed point and fixed line solutions. This is a perfectly legitimate search, since the IR limit probes small momenta, where the chiral Lagrangian is applicable. The analysis is performed as well for $SU(2) \times SU(2)$ as for $SU(3) \times SU(3)$ chiral symmetry. As it turns out, the RGE indeed exhibit IR fixed points in *ratios* of $O(p^4)$ couplings as well as IR fixed lines and surfaces relating $O(p^6)$ to $O(p^4)$ couplings and $O(p^6)$ couplings among each other. These fixed points and higher fixed manifolds are non-trivial special solutions of the renormalization group equations which do not depend on any initial values for the couplings and which in addition attract the renormalization group flow in its evolution from the scale Λ towards the IR. It is very interesting to see, how they compare to experimental data as far as these are available. This comparison is performed for the ratios of $O(p^4)$ couplings. Agreement with the fixed point values is found for a surprisingly large number of ratios, thus confirming to a certain extent the fixed point solutions by data.

A first reference to fixed points of the linearly realized chiral Lagrangian was given in Ref. [5]. For a review on the role of fixed manifolds in other contexts we refer to Ref [6]. As fixed point solutions correspond to renormalization group invariant relations between couplings, they may also be viewed as solutions of the parameter reduction program [7]. Earlier applications of the parameter reduction technique in the framework of effective field theories may be found in [8].

Let us first fix our notations for the couplings for $SU(3)$ and $SU(2)$ symmetry, respectively. The effective chiral Lagrangian for chiral $SU(3) \times SU(3)$ symmetry, expanded in terms of increasing powers of p^2 , may be written as follows

$$\mathcal{L}_\chi = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \dots = \mathcal{L}^{(2)} + \sum_i L_i \mathcal{O}_4^i + \sum_i \frac{K_i}{\Lambda^2} \mathcal{O}_6^i + \dots, \quad (1)$$

where \mathcal{O}_4 and \mathcal{O}_6 are dimension four and six operators, respectively, and where the dimensionful cut-off Λ is introduced, leading to dimensionless couplings L_i and K_i . The couplings to external right-handed and left-handed vector fields r_μ, l_μ , and scalar and pseudoscalar fields s, p are included. The lowest order, $O(p^2)$, Lagrangian $\mathcal{L}^{(2)}$ may be written as

$$\mathcal{L}_{SU(3)}^{(2)} = \frac{F_0^2}{4} \langle D_\mu U^\dagger D^\mu U \rangle + \frac{F_0^2}{4} \langle U^\dagger \chi + \chi^\dagger U \rangle. \quad (2)$$

The operation $\langle \cdot \rangle$ denotes the trace, the external fields enter through $\chi = 2B_0(s + ip)$ and the covariant derivative $D_\mu U = \partial_\mu U - ir_\mu U + iU l_\mu$. The unitary matrix U is given for $SU(3) \times SU(3)$ symmetry in terms of the Goldstone boson fields $\Phi(x)$

$$U(\Phi) = \exp\left(i\sqrt{2}\Phi/F_0\right) \quad \text{with} \quad \Phi(x) \equiv \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -2\frac{\eta}{\sqrt{6}} \end{pmatrix}. \quad (3)$$

The two constants [9] F_0, B_0 are undetermined by the symmetry; they are related to the pion decay constant and the quark condensate, respectively.

For the $O(p^4)$ Lagrangian $\mathcal{L}^{(4)}$ in the $SU(3)$ case we follow the conventions for the couplings L_i by Gasser and Leutwyler [9]

$$\begin{aligned} \mathcal{L}_{SU(3)}^{(4)} &= \sum_{i=1}^{10} L_i \mathcal{O}_i + \sum_{i=1}^2 H_i \mathcal{O}'_i \\ &= L_1 \langle D_\mu U^\dagger D^\mu U \rangle^2 + L_2 \langle D_\mu U^\dagger D_\nu U \rangle \langle D^\mu U^\dagger D^\nu U \rangle + L_3 \langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U \rangle \\ &\quad + L_4 \langle D_\mu U^\dagger D^\mu U \rangle \langle U^\dagger \chi + \chi^\dagger U \rangle + L_5 \langle D_\mu U^\dagger D^\mu U (U^\dagger \chi + \chi^\dagger U) \rangle + L_6 \langle U^\dagger \chi + \chi^\dagger U \rangle^2 \\ &\quad + L_7 \langle U^\dagger \chi - \chi^\dagger U \rangle^2 + L_8 \langle U^\dagger \chi U^\dagger \chi + \chi^\dagger U \chi^\dagger U \rangle - iL_9 \langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U \rangle \\ &\quad + L_{10} \langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu} \rangle + \text{terms involving external fields only}, \end{aligned} \quad (4)$$

where the field strength tensors of the external gauge fields are $F_{R\mu\nu} = \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu]$, and similarly for $F_{L\mu\nu}$.

For the $O(p^4)$ Lagrangian $\mathcal{L}^{(4)}$ in case of $SU(2) \times SU(2)$ symmetry we follow the conventions for the couplings l_i by Gasser and Leutwyler [10]

$$\begin{aligned} \mathcal{L}_{SU(2)}^{(4)} &= l_1 (\nabla^\mu U^\top \nabla_\mu U)^2 + l_2 (\nabla^\mu U^\top \nabla^\nu U) (\nabla_\mu U^\top \nabla_\nu U) + l_3 (\chi^\top U)^2 + l_4 (\nabla^\mu \chi^\top \nabla_\mu U) + \\ &\quad + l_5 (U^\top F^{\mu\nu} F_{\mu\nu} U) + l_6 (\nabla^\mu U^\top F_{\mu\nu} \nabla^\nu U) + l_7 (\tilde{\chi}^\top U)^2 + \text{terms involving external fields}, \end{aligned} \quad (6)$$

where $U^A(x)$ is a four-component real $O(4)$ vector field of unit length, $U^\top U = 1$, with covariant derivative

$$\nabla_\mu U^0 = \partial_\mu U^0 + a_\mu^i(x) U^i, \quad \nabla_\mu U^i = \partial_\mu U^i + \epsilon^{ikl} v_\mu^k(x) U^l - a_\mu^i(x) U^0. \quad (7)$$

The external fields are $v_\mu(x) = v_\mu^i(x) \tau^i / 2$, $a_\mu(x) = a_\mu^i(x) \tau^i / 2$, $s(x) = s^0(x) 1 + s^i(x) \tau^i$ and $p(x) = p^0(x) 1 + p^i(x) \tau^i$ and the vectors χ and $\tilde{\chi}$ are given by $\chi^A = 2B(s^0, p^i)$ and $\tilde{\chi}^A = 2B(p^0, -s^i)$.

The $O(p^6)$ couplings will be denoted by K_i for $SU(3)$ and by k_i for $SU(2)$. There are too many to be spelt out here, they may be found in Ref. [3].

Following the renormalization procedure for effective theories outlined in the introduction, we use dimensional regularization and a mass independent renormalization scheme. According to the renormalization program for the chiral Lagrangian [1], the relations between bare and renormalized couplings are:

$$SU(3): \quad L_i^b = \mu^{(d-4)} [L_i^r + \Gamma_i \lambda], \quad (8)$$

$$K_i^b = \mu^{2(d-4)} \left[K_i^r + (C_i + D_{ij} L_j^r) \lambda + A_i \lambda^2 \right], \quad (9)$$

$$SU(2) : \quad l_i^b = \mu^{(d-4)} [l_i^r + \gamma_i \lambda], \quad (10)$$

$$k_i^b = \mu^{2(d-4)} \left[k_i^r + (c_i + d_{ij} l_j^r) \lambda + a_i \lambda^2 \right], \quad (11)$$

where $\lambda = [(d-4)^{-1} - \zeta]/16\pi^2$, and ζ is a constant which depends on the renormalization scheme. The constants Γ_i , C_i , D_{ij} , A_i for $SU(3)$ and γ_i , c_i , d_{ij} , a_i for $SU(2)$ are calculable and known; they are scheme independent. The constants Γ_i , γ_i will be summarized below, for the others we refer to Ref. [3].

Unless a distinction between the $SU(2)$ and the $SU(3)$ cases is necessary, in the following the capital letters L_i , K_i , Γ_i , C_i , D_i are used as common fill-ins for the $SU(3)$ quantities and for the $SU(2)$ quantities l_i , k_i , γ_i , c_i , d_i .

The renormalization group equations (RGE) for the renormalized couplings then have the following generic form

$$\mu \frac{dL_i^r}{d\mu} = -\frac{1}{16\pi^2} \Gamma_i, \quad (12)$$

$$\mu \frac{dK_i^r}{d\mu} = -\frac{1}{16\pi^2} (C_i + D_{ij} L_j^r), \quad (13)$$

where summation over repeated indices is implied. Henceforth we shall drop the index r , since we shall exclusively deal with renormalized quantities.

Let us next proceed with an analysis of the renormalization group equations (12) for the $O(p^4)$ couplings L_i for infrared fixed points and their comparison to experiment. It is obvious that Eq. (12) has no IR fixed point. However, the RGE for the ratios L_i/L_j of couplings

$$\mu \frac{d}{d\mu} \left(\frac{L_i}{L_j} \right) = \frac{1}{16\pi^2} \frac{\Gamma_j}{L_j} \left(\frac{L_i}{L_j} - \frac{\Gamma_i}{\Gamma_j} \right) \quad (14)$$

have a non-trivial fixed point each at

$$\left. \frac{L_i}{L_j} \right|_{f.p.} = \frac{\Gamma_i}{\Gamma_j} \quad \text{for} \quad \Gamma_i, \Gamma_j \neq 0. \quad (15)$$

The general solution of the RGE (12) for initial value $L_i(\Lambda)$ at $\mu = \Lambda$ is

$$L_i(\mu) = L_i(\Lambda) - \frac{1}{16\pi^2} \Gamma_i \log \frac{\mu}{\Lambda}. \quad (16)$$

Obviously, in the IR limit $\mu \rightarrow 0$ the ratios $L_i(\mu)/L_j(\mu)$ of *all general solutions* approach the corresponding IR fixed points (15), controlled by the approach of $\log(\mu/\Lambda) \rightarrow -\infty$.

It is now interesting to compare the experimental values for the ratios of couplings with the predictions obtained for their IR fixed points (15).

The following tables summarize the experimental values of the couplings l_i for $SU(2)$ symmetry [9, 10] at an energy scale of the order of the pion mass and of the couplings L_i for $SU(3)$ symmetry [11]

at an energy scale $\mu = m_\rho$ (which is unfortunately rather large). These values are derived from meson decay constants, electromagnetic form factors and, for $SU(3)$ symmetry, also from semileptonic kaon decays. Also the corresponding coefficients γ_i resp. Γ_i of the beta functions are given.

$SU(2):$	i	1	2	3	4	5	6
	$l_i \cdot 10^3$	-2.4 ± 3.9	12.7 ± 2.7	-4.6 ± 3.8	27.2 ± 5.7	-7.3 ± 0.7	-17.4 ± 1.2
	γ_i	1/3	2/3	-1/2	2	-1/6	-1/3

$SU(3):$	i	1	2	3	4	5
	$L_i \cdot 10^3$	0.4 ± 0.3	1.35 ± 0.3	-3.5 ± 1.1	-0.3 ± 0.5	1.4 ± 0.5
	Γ_i	3/32	3/16	0	1/8	3/8

	i	6	7	8	9	10
	$L_i \cdot 10^3$	-0.2 ± 0.3	-0.4 ± 0.2	0.9 ± 0.3	6.9 ± 0.7	-5.5 ± 0.7
	Γ_i	11/144	0	5/48	1/4	-1/4

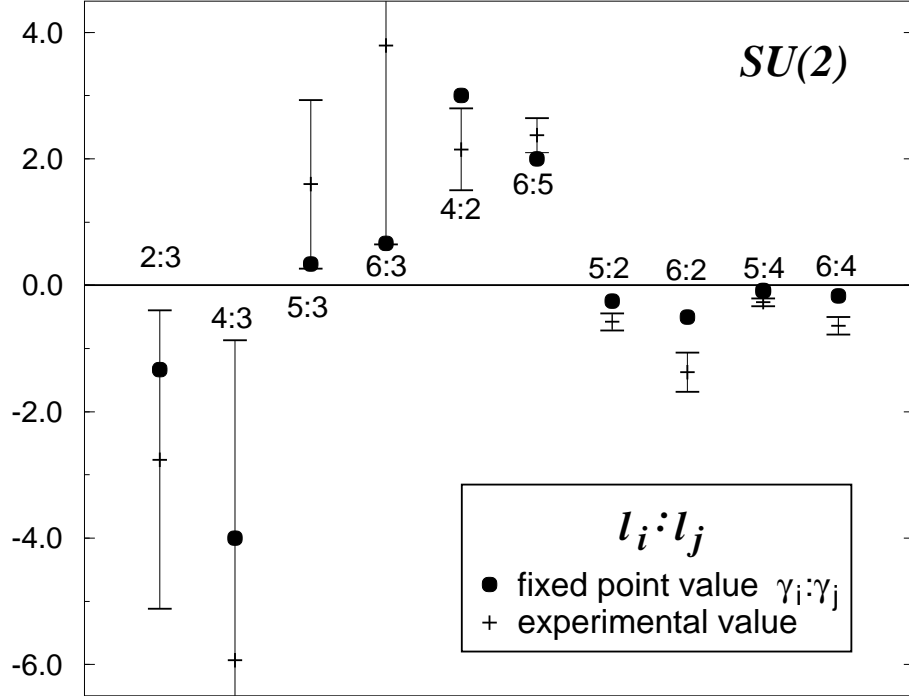


Figure 1: Ratios l_i/l_j of the experimental values of the couplings l_i in comparison with the corresponding IR fixed point ratios γ_i/γ_j for $SU(2)$ symmetry. To guide the eye, the ratios are ordered from left to right according to decreasing agreement.

In Figs. 1 and 2 the experimental ratios l_i/l_j for $SU(2)$ and L_i/L_j for $SU(3)$ are compared with their respective fixed point ratios γ_i/γ_j and Γ_i/Γ_j for values of i, j for which $\gamma_i, \gamma_j, \Gamma_i$ and Γ_j are nonzero. We omitted l_1 for $SU(2)$ and L_4 and L_6 for $SU(3)$, since their errors are too large to give meaningful comparisons. To guide the eye, we ordered the ratios from left to right with

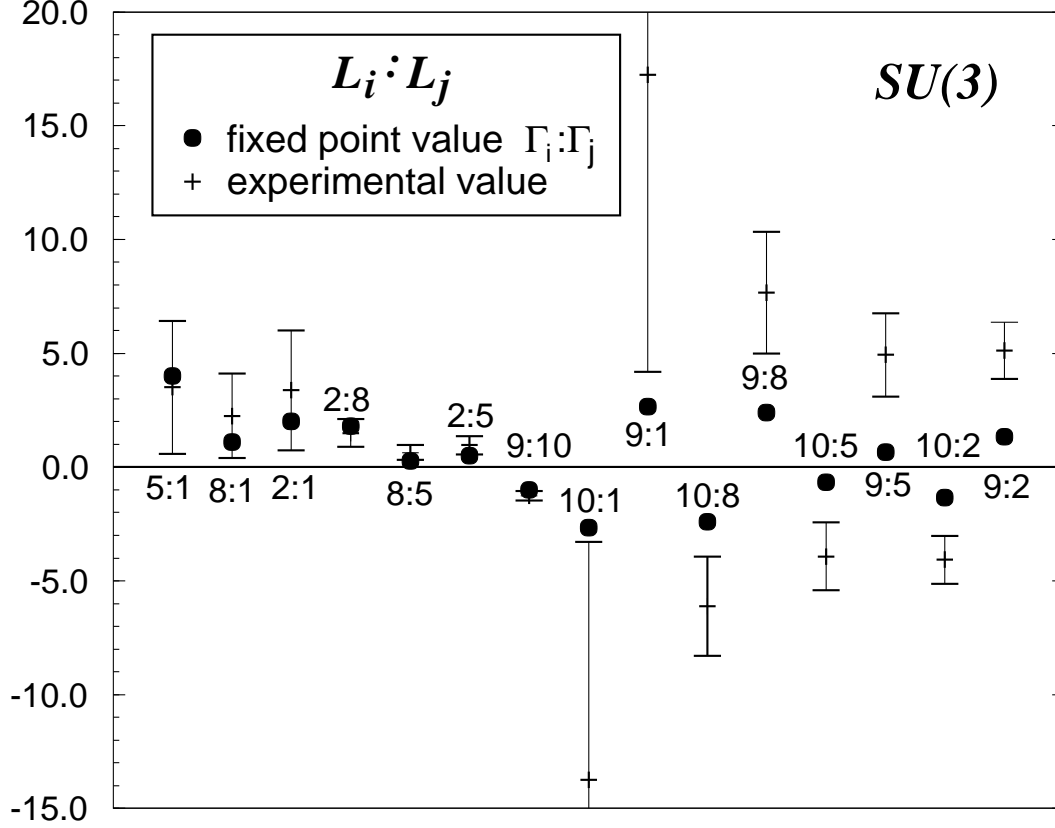


Figure 2: Ratios L_i/L_j of the experimental values of the couplings L_i in comparison with the corresponding IR fixed point ratios Γ_i/Γ_j for $SU(3)$ symmetry. To guide the eye, the ratios are ordered from left to right according to decreasing agreement.

decreasing agreement of experimental values with the fixed point values. The overall agreement between experimental and predicted values is quite impressive. Even where the agreement fails on a quantitative level (at the right end of the figures), the sign and the qualitative tendency are correct and the deviation is generically of the order of one standard deviation.

Encouraged by this amount of experimental support, we proceed to the analysis of the $O(p^6)$ RGE (13). The general solution of the generic RGE (13) for the $O(p^6)$ couplings $K_i(\mu)$ in terms of the general solution for the $O(p^4)$ couplings $L_i(\mu)$ is

$$K_i(\mu) = K_i(\Lambda) + \frac{1}{2D_{ij}\Gamma_j} \left((C_i + D_{ij}L_j(\mu))^2 - (C_i + D_{ij}L_j(\Lambda))^2 \right). \quad (17)$$

The general solution as function of μ follows by inserting the general solution (16) for L_i into Eq. (17)

$$K_i(\mu) = K_i(\Lambda) - \frac{1}{16\pi^2} (C_i + D_{ij}L_j(\Lambda)) \log \frac{\mu}{\Lambda} + \frac{1}{2(16\pi^2)^2} D_{ij}\Gamma_j \left(\log \frac{\mu}{\Lambda} \right)^2. \quad (18)$$

First of all we notice a renormalization group invariant relation between the $O(p^6)$ couplings K_i

and the $O(p^4)$ couplings L_j , an *IR attractive fixed line* in the plane of K_i versus $D_{ij}L_j$

$$K_i = \frac{1}{2D_{ij}\Gamma_j} (C_i + D_{ij}L_j)^2, \quad \text{valid for all values of } \mu. \quad (19)$$

(or a fixed point in the variable $K_i/(C_i + D_{ij}L_j)^2$) or an infrared attractive fixed hypersurface in the space of K_i versus the contributing variables L_j . This special solution of the RGE (12,13) is not determined by initial value conditions; it is *IR attractive for all solutions* (16),(17), i.e. for the renormalization group flow, in the limit $\mu \rightarrow 0$, i.e. $\log(\mu/\Lambda) \rightarrow -\infty$. This is a very interesting result, which predicts the K_i in terms of the L_i .

If one inserts furthermore the fixed point solutions (15), which implies $D_{ij}L_j = D_{ij}\Gamma_j(L_m/\Gamma_m)$ for any suitable fixed value of m , into the fixed line solution (19), one obtains by elimination of the coupling L_m a renormalization group invariant relation between any two $O(p^6)$ couplings K_i and K_j

$$\sqrt{K_i} - \frac{C_i}{\sqrt{2D_{ik}\Gamma_k}} = \sqrt{\frac{D_{ik}\Gamma_k}{D_{jk}\Gamma_k}} \left(\sqrt{K_j} - \frac{C_j}{\sqrt{2D_{jk}\Gamma_k}} \right). \quad (20)$$

Again this fixed line in the plane K_i versus K_j is IR attractive for the RG flow. In the IR limit $\mu \rightarrow 0$, this fixed line approaches a constant ratio for the ratios of the $O(p^6)$ couplings

$$\frac{K_i}{K_j} \rightarrow \frac{D_{ik}\Gamma_k}{D_{jk}\Gamma_k} \quad \text{for } \mu \rightarrow 0. \quad (21)$$

The IR fixed manifolds (19) and (20) are solutions establishing renormalization group invariant relations between couplings independently of any initial values. They may be similarly relevant in nature as the IR fixed points (15) for ratios of $O(p^4)$ couplings which found substantial support by data. In addition the fixed manifolds (19) and (20) are IR attractive for the renormalization group flow in its evolution from $\mu = \Lambda$ towards the IR. Even though this evolution path is in practice not very long, let us illustrate in Fig.3 the mathematical fact of this IR attraction (for a fictitious generic case $\Gamma_{1,2} = 1$, $C_{1,2} = 1/(16\pi^2)$, $D_{11} = D_{12} = D_{21} = 1$, $D_{22} = 2$ and suitable initial values). For presentation reasons the dependent $O(p^6)$ variables K are normalized such that their IR fixed value is equal to one.

It has to be realized that if it were not for accidental zeros, the IR fixed manifolds express all $O(p^4)$ and $O(p^6)$ in terms of a single coupling. In practice, it will be a comparatively small number of independent couplings.

The relations (12)-(21) have been derived in the $SU(3)$ nomenclature with capital letters. As pointed out earlier, they are valid also for $SU(2)$ symmetry, if the capital letters are replaced by the small ones. The relations hold for any of the relevant sets of couplings L_i , K_j and l_i , k_j with the numbers Γ_i and γ_i as given in the tables and the numbers C_i , D_{ij} and c_i , d_{ij} as given in Ref. [3]. We found reasonable experimental support for the infrared fixed point solutions for ratios of $O(p^4)$ couplings. There is, unfortunately, not sufficient experimental information to subject the infrared fixed line solutions between $O(p^4)$ and $O(p^6)$ couplings and among $O(p^6)$ couplings to a similar test.

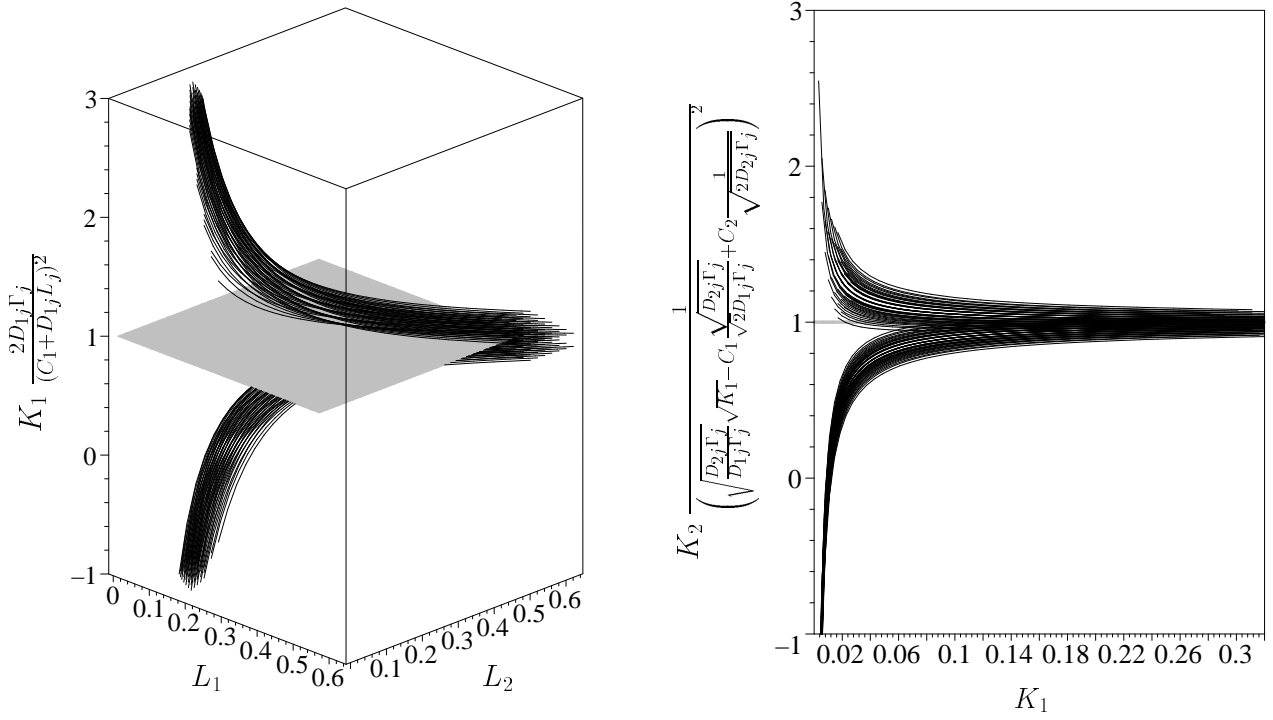


Figure 3: Generic example, illustrating the IR attraction of the renormalization group flow towards the IR fixed manifolds which relate $O(p^6)$ couplings to $O(p^4)$ couplings and $O(p^6)$ couplings among each other.

Acknowledgements. One of us (M.A.) would like to thank the Institut für Theoretische Physik und Astrophysik der Universität zu Kiel for hospitality, and the spanish MEC for a grant of the FPU programme. One of us (B.S.) is grateful to the theory group of DESY for continuous hospitality.

References

- [1] S. Weinberg, *Physica* **96A** (1979) 327
- [2] See e.g., A. Pich, “Effective Field Theory: Course”, Lectures at the 1997 Les Houches Summer School “Probing the Standard Model of Particle Interactions”, [hep-ph/9806303](#) and references quoted therein
- [3] J. Bijnens, G. Colangelo and G. Ecker, “Renormalization of Chiral Perturbation Theory to Order p^6 ”, [hep-ph/9907333](#), *Annals Phys.* **280** (2000) 100 and references quoted therein

- [4] For reviews see J. Bijnens, “Goldstone Boson Production and Decay”, [hep-ph/9710341](#); J. Bijnens and U. G. Meissner, “Chiral Effective Theories”, [hep-ph/9901381](#) and references quoted therein
- [5] D. U. Jungnickel and C. Wetterich, *Phys. Rev. D* **53** (1996) 5142
- [6] B. Schrempp and M. Wimmer, “Top Quark and Higgs Boson Masses: Interplay between Infrared and Ultraviolet Physics”, *Progress in Particle and Nuclear Physics* **37** (1996) 1 and [hep-ph/9606386](#)
- [7] see e.g. J. Kubo, “Applications of the Reduction of Couplings”, [hep-ph/9903482](#) and R. Oehme, “Reduction of Coupling Parameters and Duality”, [hep-th/9903092](#)
- [8] M. Atance and J. L. Cortés, *Phys. Lett. B* **387** (1996) 697; *Phys. Rev. D* **54** (1996) 4973, *D* **56** (1997) 3611; L. A. Toro, *Z. Phys. C* **56** (1992) 635
- [9] J. Gasser and H. Leutwyler, *Nucl. Phys. B* **250** (1985) 465
- [10] J. Gasser and H. Leutwyler, *Ann. Phys. NY* 158 (1984) 142
- [11] J. Bijnens, G. Ecker and J. Gasser, “Chiral Perturbation Theory”, in *The DAΦNE Physics Handbook* (second edition), eds. L. Maiani, G. Pancheri and N. Paver (Frascati, 1994)